Vidya Bhawan, Balika Vidyapith

Shakti Utthan Ashram, Lakhisarai – 811311 (Bihar)

Class: -X *Topic: - Polynomial* **Subject: -Mathematics**

Graph of Quadratic Polynomial

Graph of Quadratic Polynomial $ax^2 + bx + c$

$P(x) = ax^2 + bx + c$

$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$= a\left[x^2 + 2x\,\frac{b}{2a} + \frac{c}{a}\right]$

$= a\left[x^2 + 2x\,\frac{b}{2a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$

(Adding and subtracting $\frac{b^2}{4a^2}$)

$= a\left[\left(x + \frac{b}{2a}\right) - \frac{b^2}{4a^2} + \frac{c}{a}\right]$ (let $k = \frac{-b^2}{4a^2} + \frac{c}{a}$)

$= a\left[\left(x + \frac{b}{2a}\right)^2 + k\right]$

Now Zeroes $p(x) = 0$

$= a\left[\left(x + \frac{b}{2a}\right)^2 + k\right] = 0$

$\left(x + \frac{b}{2a}\right)^2 = 0$ or $k = 0$

$\therefore \left(x + \frac{b}{2a}\right) = 0$ $\therefore x = \frac{-b}{2a}$

Let $p(x) = y$

Given $P(x) = y = ax^2 + bx + c$

Putting $x = \frac{-b}{2a}$

$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = \left[\frac{ab^2}{4a^2} - \frac{b^2}{2a} + c\right]$

$= \left[\frac{ab^2 - 2ab^2 + 4a^2c}{4a^2}\right]$

$= -a\left(\frac{b^2 - 4ac}{4a^2}\right) = -\left(\frac{b^2 - 4ac}{4a}\right) = \frac{-D}{4a}$.

(where $D = b^2 - 4ac$)

$\therefore y = \frac{-D}{4a}$

Hence $(x, y) = \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

Now Y- intercept

$y = ax^2 + bx + c$ {coordinate of Y-axis = (0, y)}

$y = a(0)^2 + b(0) + c = c$

Y- intercept = (0, c) =(0, constant)

Q. Draw the graph of the polynomial

$p(x) = x^2 - 2x - 8$

Solution: - $p(x) = x^2 - 2x - 8$

1. The value of a = 1 is positive than parabola opens upward

$p(x) = x^2 - 2x - 8 = x^2 - 4x + 2x - 8$

$= x(x - 4) + 2(x - 4) = (x - 4)(x + 2)$

$p(x) = 0$

$\therefore (x - 4) = 0$ or $(x + 2) = 0$

$\therefore x = 4$ or $x = -2$

2. The parabola cuts X –axis at two distinct points (4, 0) and (-2, 0)

3. Comparing $ax^2 + bx + c = x^2 - 2x - 8$;
a=1, b = -2, c = - 8

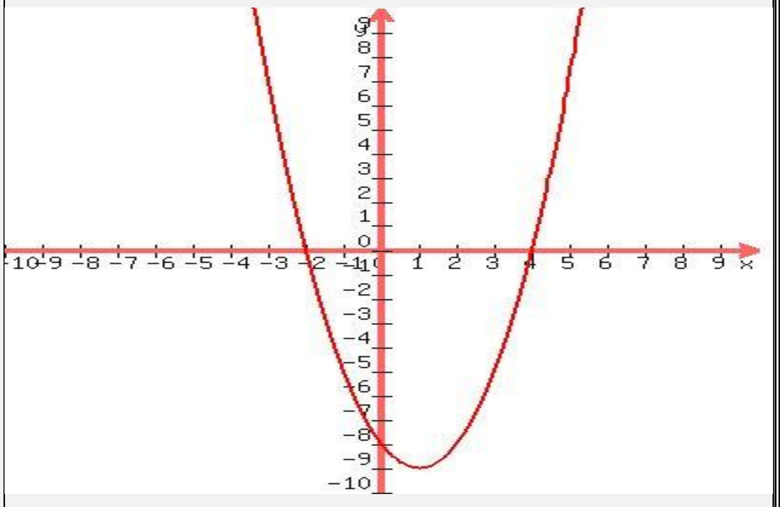
$x = \frac{-b}{2a} = \frac{-(-2)}{2\times1} = \frac{2}{2} = 1$

$y = \frac{-D}{4a} = \frac{-(b^2 - 4ac)}{4\times1} = -\frac{[(-2)^2 - 4\times1\times-8]}{4}$

$= -\left[\frac{4+32}{4}\right] = -9$

4. Vertex of Parabola $(x, y) = \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ = (1, -9)

Y- intercept = (0, c) =(0, constant) = (0, -8)



Cubic Polynomial: A polynomial having highest degree of three is called a cubic polynomial. In general, a quadratic polynomial can be expressed in the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c, d are real numbers.

$P(x) = 3x^3 - x^2 - 3x + 5$, $P(x) = 3x^3 - 3x + 5$, $P(x) = 3x^3 + 5$

Zeroes Cubic Polynomial: In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then,

Coefficient of x^3 = a, Coefficient of x^2 = b, Coefficient of x = c, Coefficient of x^0 (constant) = d

RELATION BETWEEN COEFFICIENT AND ZEROES

Sum of Zeroes $= \alpha + \beta + \gamma = \dfrac{-\text{Coefficient of } x^2}{\text{Coefficient of } x^3} = \dfrac{-b}{a}$

Sum of the products of zeroes taken two at a time $= \alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = \dfrac{c}{a}$

Product of zeroes $\alpha\beta\gamma = \dfrac{-\text{constant}}{\text{Coefficient of } x^3} = \dfrac{-d}{a}$

Proof: - if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then,

$ax^3 + bx^2 + cx + d = k(x - \alpha)(x - \beta)(x - \gamma)$

$a\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right)$

$= k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - (\alpha\beta\gamma)]$

Comparing the terms according to exponent

Therefore $a = k$, $x^3 = x^3$,

$\frac{b}{a}x^2 = -(\alpha + \beta + \gamma)x^2 \Rightarrow \alpha + \beta + \gamma = \frac{-b}{a}$

$$\frac{c}{a}x = (\alpha\beta + \beta\gamma + \gamma\alpha)x \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\frac{d}{a} = -(\alpha\beta\gamma) \Rightarrow \alpha\beta\gamma = \frac{-d}{a}$$

If α, β, γ are the zeroes, then cubic polynomial

$= k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - (\alpha\beta\gamma)]$

=k[x^3 – (Sum of Zeroes)x^2 +

(Sum of the products of zeroes taken two at a time)x –

(Product of zeroes)]

Q. Verify that $-1, \frac{-1}{3}, 3$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution: $p(x) = 3x^3 - 5x^2 - 11x - 3$

If $-1, \frac{-1}{3}$, 3 are zeroes of p(x) then leaves remainder p(x) = 0

$p(x) = 3x^3 - 5x^2 - 11x - 3$

$p(-1) = 3(-1)^3 - 5(-1)^2 - 11(-1) - 3$ Putting x = -1

$= -3 - 5 + 11 - 3 = 11 - 11 = 0$

$p(\frac{-1}{3}) = 3(\frac{-1}{3})^3 - 5(\frac{-1}{3})^2 - 11(\frac{-1}{3}) - 3$ Putting $x = \frac{-1}{3}$

$= \frac{-1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = \frac{-1 - 5 + 33 - 27}{9} = \frac{33 - 33}{9} = 0$

$p(3) = 3(3)^3 - 5(3)^2 - 11(3) - 3$ Putting x = 1

$= 81 - 45 - 33 - 3 = 81 - 81 = 0$

$\therefore -1, \frac{-1}{3}$ & 3 are the zeroes of p(x) *Answer*

Verify the relationship between the zeroes and the coefficients.

$ax^3 + bx^2 + cx + d = 3x^3 - 5x^2 - 11x - 3$

a = 3, b = -5, c = -11 & d = -3

Let $\alpha = -1$, $\beta = \frac{-1}{3}$ & $\gamma = 3$

$\alpha + \beta + \gamma = \frac{-b}{a} \Rightarrow -1 + (\frac{-1}{3}) + 3 = \frac{-3-1+9}{3} = \frac{5}{3}$

$\therefore \frac{5}{3} = \frac{5}{3}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \Rightarrow -1 \times \frac{-1}{3} + \left(\frac{-1}{3}\right) \times 3 + 3 \times -1$

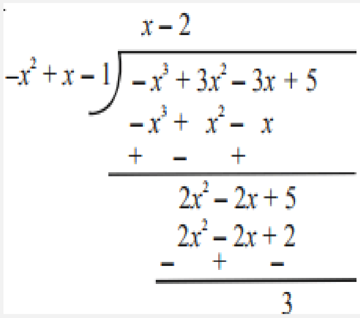
$= \frac{1}{3} + (-1) - 3 = \frac{-11}{3}$ $\therefore \frac{-11}{3} = \frac{-11}{3}$

$\alpha\beta\gamma = \frac{-d}{a} \Rightarrow -1 \times (\frac{-1}{3}) \times 3 = \frac{3}{3}$ $\therefore 1 = 1$ *Verified*

Division Algorithm for Polynomials : If p(x) and g(x) are any two polynomials with $g(x) \neq 0$, then we can find polynomials q(x) and r(x) such that $p(x) = g(x) \times q(x) + r(x)$, where $r(x) = 0$ or degree of r(x) < degree of g(x).

Example: Divide $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, and verify the division algorithm.

On dividing $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, we get,



∴ Quotient is $(x - 2)$ and remainder is 3. *Answer*

Now, as per the division algorithm

Divisor x Quotient + Remainder = Dividend

Let $p(x) = 3x^2 - x^3 - 3x + 5$,

$g(x) = -x^2 + x - 1$, $q(x) = x - 2$ & $r(x) = 3$

$p(x) = g(x) \times q(x) + r(x)$,

$3x^2 - x^3 - 3x + 5 = (-x^2 + x - 1)(x - 2) + 3$

$= (-x^3 + x^2 - x + 2x^2 - 2x + 2 + 3)$

$= (-x^3 + 3x^2 - 3x + 5)$

$(-x^3 + 3x^2 - 3x + 5) = (-x^3 + 3x^2 - 3x + 5)$

∴ LHS = RHS

∴ Division algorithm is verified. *Verified*

Q. Obtain all the zeroes of $3x^4+6x^3-2x^2-10x-5$ if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

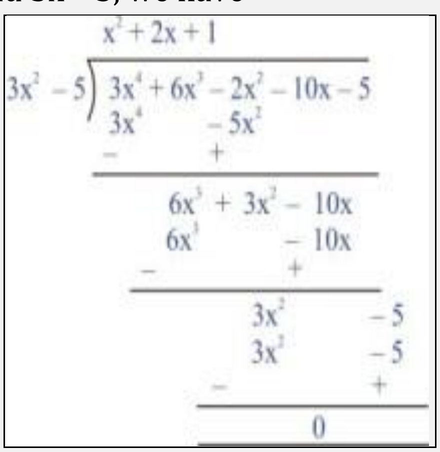
Since two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$ so these are the factors of the given polynomial.

Now, $(x - \sqrt{\frac{5}{3}})(x + \sqrt{\frac{5}{3}}) = x^2 - \frac{5}{3}$

[Rough $x^2 - \frac{5}{3} = 0$, $3x^2 - 5 = 0$]

⇒ $(3x^2-5)$ is a factor of the given polynomial.

Applying the division algorithm to the given polynomial and $3x^2-5$, we have



∴ $3x^4+6x^3-2x^2-10x-5 = (3x^2-5)(x^2+2x+1) + 0$

Now, $x^2+2x+1 = x^2+x+x+1$

$= x(x+1)+1(x+1) = (x+1)(x+1)$

So, its other zeroes are -1 and -1.

Thus, all the zeroes of the given fourth degree polynomial are, $-\sqrt{\frac{5}{3}}$, $\sqrt{\frac{5}{3}}$ -1 and -1. *Answer*

Q. On dividing x^3-3x^2+x+2 by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$ respectively. Find $g(x)$.

Solution. Since on dividing $x^3 - 3x^2 + x + 2$ by a polynomial $(x - 2) \times g(x)$,

The quotient and remainder were $(x - 2)$ and $(-2x + 4)$ respectively,

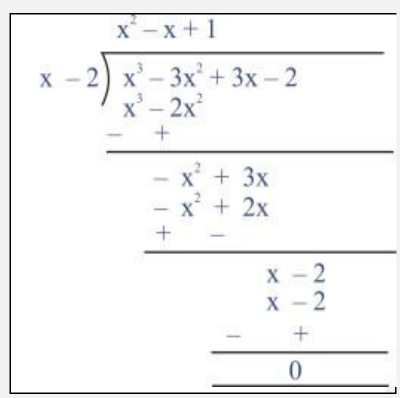
Quotient × Divisor + Remainder = Dividend

⇒ $(x-2) \times g(x) + (-2x + 4) = x^3 - 3x^2 + x + 2$

⇒ $(x - 2) \times g(x) = x^3 - 3x^2 + x + 2 + 2x - 4$

⇒ $g(x) = x^3 - 3x^2 + 3x - 2 / (x - 2)$ ... (1)

Let us divide $x^3 - 3x^2 + 3x - 2$ by $x-2$. We get



⇒ $g(x) = x^3 - 3x^2 + 3x - 2 / (x - 2)$

⇒ $g(x) = (x - 2)(x^2-x+1) / (x - 2)$

∴ equation (1) gives $g(x) = x^2-x+1$ *Answer*

Q. Give an example of polynomial p(x), g(x), q (x) and r (x), which satisfy the division algorithm and deg p(x) = deg q(x).

There can be several examples for

deg p(x) = deg q(x) which satisfy the division algorithm

Example: -$p(x)=2x^2-2x+19$, $g(x)=2$, $q(x)=x^2-x+7$, $r(x)=5$

By division algorithm

$p(x) = g(x) \times q(x) + r(x)$,

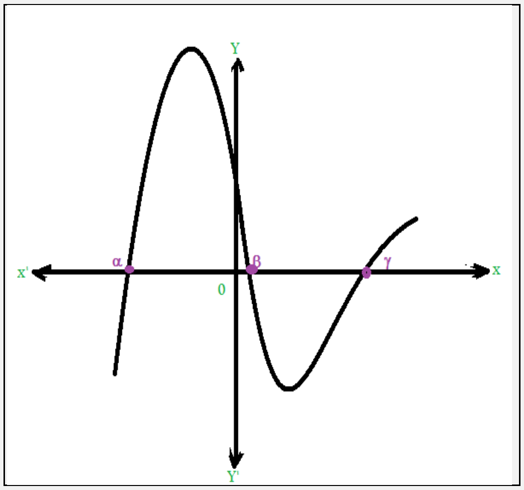
$2x^2-2x+19 = 2 \times (x^2-x+7) + 5$

$= 2x^2-2x+14 + 5$

$2x^2-2x+19 = 2x^2-2x+19$

Satisfied division algorithm **and also satisfied degree of p(x) =degree of q(x) = 2** *Answer*

Graph of Cubic Polynomial



The graph of cubic polynomial cuts x-axis in three times which is denoted by α, β and γ.

SUBJECT ENRICHMENT

Draw Graph of Given Quadratic Polynomials

(a) $x^2 - 2x - 8$ **(b) $-x^2 - 2x + 3$**

(c) $3x^2 - 6x + 3$ **(d) $-4x^2 + 4x - 1$**

(e) $3x^2 + 2x + 1$ **(f) $-3x^2 + 2x - 2$**

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